Recall Taylor's Theorem

Let $I = [a,b]$ and $f: I \to \mathbb{R}$ be an function with $f', f'', \dots, f^{(n+1)}$ exists. Then for any $x_0, x \in I$, there exists c between x_0 and x s.t.

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

Rmk: Taylor's Theorem can be viewed as a generalization of MVT.

Precisely, MVT is Taylor's Theorem with $n = 0$.

1. Approximate the number e with error less than $\frac{1}{10}$

pf: Let $f(x) = e^x$, $x_0 = 0$, $x = 1$

Note that $f^{(n)}(x) = e^x$ for any $n \in \mathbb{N}$.

By Taylor's Theorem, for any $n \in \mathbb{N}$, there exists $c \in (0, 1)$ s.t.

$$e = f(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(x_0)}{(n+1)!}(x - x_0)^{n+1}$$

$$= 1 + 1 + \cdots + \frac{1}{n!} + \frac{e^c}{(n+1)!}$$

Since $0 < c < 1$, then $1 < e^c < e^1 < 3$.

Take $n = 4$, then $0 < \frac{e^c}{(n+1)!} < \frac{3}{5!} = \frac{1}{40} < \frac{1}{10}$.

$$e \approx 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} = 2 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24}$$

$$= \frac{65}{24}$$

with error less than $\frac{1}{10}$.

2 Recall $e^x > 1+x$ for any $x > 0$.

In general, $e^x > 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$ for any $x > 0$.

Pf: Let $f(x) = e^x$ and $x_0 = 0$.

By Taylor's Theorem, for any $x > 0$,

there exists some $c \in (0, x)$ s.t.

$$e^x = f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

$$= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{e^c}{(n+1)!}x^{n+1}$$

$$> 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

3. $\cos x \geq 1 - \frac{x^2}{2}$ for $x \in \mathbb{R}$.

Pf: Let $f(x) = \cos x$, $x_0 = 0$, $n = 2$

Note that $f'(x) = -\sin x$

$$f''(x) = -\cos x$$

$$f'''(x) = \sin x$$

By Taylor's Theorem, for any $x \in \mathbb{R}$, there exists c between x and 0 s.t.

$$\cos x = f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(c)}{3!}(x - x_0)^3$$

$$= 1 + 0 - \frac{1}{2}x^2 + \frac{\sin c}{6}x^3$$

- If $0 \leq x \leq \pi$, $0 \leq c \leq \pi$.

  Then $\sin c \geq 0$ and $x^3 \geq 0$.

  Thus $\frac{\sin c}{6}x^3 \geq 0$ and $\cos x \geq 1 - \frac{x^2}{2}$

- If $-\pi \leq x \leq 0$, $-\pi \leq c \leq 0$.

  Then $\sin c \leq 0$ and $x^3 \leq 0$.

  Thus $\frac{\sin c}{6}x^3 \geq 0$ and $\cos x \geq 1 - \frac{x^2}{2}$

- If $|x| > \pi$, then

$$1 - x^2 < 1 - \pi^2 < 1 - 2^2 = -3 < \cos x$$

$\square$

4. $$x - \frac{x^2}{2} + \cdots - \frac{x^{2k}}{2k} < \ln(x+1) < x - \frac{x^2}{2} + \cdots - \frac{x^{2k}}{2k} + \frac{x^{2k+1}}{2k+1}$$

for any $x > 0$.

Pf: Let $f(x) = \ln(x+1)$ and $x_0 = 0$

Then $f'(x) = \frac{1}{x+1}$

$f''(x) = -\frac{1}{(x+1)^2}$

$f'''(x) = \frac{2}{(x+1)^3}$

Claim: $f^{(n)}(x) = (-1)^{n-1} \frac{(n-1)!}{(x+1)^n}$ for any $n \in \mathbb{N}$.

Pf by induction: Suppose it holds for m

$$f^{(m+1)}(x) = \left(f^{(m)}(x)\right)' = (-1)^{m-1}(m-1)!\left(\frac{1}{(x+1)^m}\right)'$$

$$= (-1)^{m-1}(m-1)! \frac{-m}{(x+1)^{m+1}}$$

$$= (-1)^m \frac{m!}{(x+1)^{m+1}}$$

Thus $\frac{f^{n}(x)}{n!} = \frac{(-1)^{n-1}}{n(x+1)^n}$ and $\frac{f^{n}(x_0)}{n!} = \frac{(-1)^{n-1}}{n}$

- Take $n=2k$. By Taylor's Theorem, for any $x>0$, there exists $c \in (0, x)$ s.t.

$$\ln(x+1) = f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(2k)}(x_0)}{(2k)!}(x-x_0)^{2k} + \frac{f^{(2k+1)}(c)}{(2k+1)!}(x-x_0)^{2k+1}$$

$$= 0 + x - \frac{x^2}{2} + \cdots - \frac{x^{2k}}{2k} + \frac{x^{2k+1}}{(2k+1)(c+1)^{2k+1}}$$

$$> x - \frac{x^2}{2} + \cdots - \frac{x^{2k}}{2k}$$

- Take $n=2k+1$. By Taylor's Theorem, for any $x>0$, there exists $c \in (0, x)$ s.t.

$$\ln(x+1) = f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(2k)}(x_0)}{(2k)!}(x-x_0)^{2k} + \frac{f^{(2k+1)}(x_0)}{(2k+1)!}(x-x_0)^{2k+1} + \frac{f^{(2k+2)}(c)}{(2k+2)!}(x-x_0)^{2k+2}$$

$$= 0 + x - \frac{x^2}{2} + \cdots - \frac{x^{2k}}{2k} + \frac{x^{2k+1}}{2k+1} - \frac{x^{2k+2}}{(2k+2)(c+1)^{2k+2}}$$

$$< x - \frac{x^2}{2} + \cdots - \frac{x^{2k}}{2k} + \frac{x^{2k+1}}{2k+1}$$

$\square$